Module - 4 - COMPLEX INTEGRATION

LINE INTEGRALS IN A COMPLEX PLANE

Complex definite Integrals are called complex line integrals written as $\int_C f(z)dz$ where C is the path of integration. We may represent the curve C by a parametric representation

$$z(t) = x(t) + i\,y(t) \qquad a \le t \le b.$$

If C is a closed path then $\int_C f(z)dz$ is represented as $\oint_C f(z)dz$

Note: All paths of integration for complex line integrals are assumed to be piecewise smooth. that is, they consist of finitely many smooth curves joined end to end.

BASIC PROPERTIES

1. Linearity

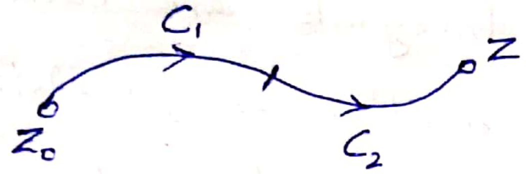
$$\int_C [k_1 f(z) + k_2 g(z)]dz = k_1 \int_C f(z)dz + k_2 \int_C g(z)dz$$

2. Sense reversal

$$\int_{z_0}^{z} f(z)dz = -\int_{z}^{z_0} f(z)dz$$

3. Partitioning of path

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



Simple closed curve

It is a simple curve that doesn't intersect

Eg circle, ellipse.

Simply connected domain

A domain D is said to be simply connected if every simple closed curve encloses only points of D.

~~Simply~~ Eg: closed circular disk

closed annulus is not simply connected.

First Evaluation Method

Indefinite Integration of Analytic functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D that is an analytic function $F(z)$ such that $F'(z) = f(z)$ in D and for all paths in D joining two points z_0 and z_1 in D we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0)$$

$F'(z) = f(z)$

SECOND EVALUATION METHOD (Use of a representation of a path)

Integration by the use of a path

Let C be a piecewise smooth path, represented by $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,\dot{z}(t)\,dt$$

$$\dot{z} = \frac{dz}{dt}$$

Eg: for first evaluation method.

$$\int_0^{1+i} z^2\,dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{-2}{3} + \frac{2}{3}i$$

Parametric Representation $z(t) = x(t) + i\,y(t)$

I Find the parametric representation of the following ~~points~~

1. $y = 1 - \frac{x^2}{2}$ and $-2 \le x \le 2$

put $x = t$

$y = 1 - \frac{t^2}{2}$

$z(t) = x(t) + i\,y(t)$

$z(t) = t + i\left[1 - \frac{t^2}{2}\right] \quad -2 \le t \le 2$

2. Line segment from $(-1, 2)$ to $(1, 4)$; $-1 \le x \le 1$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \Longrightarrow \frac{x+1}{2} = \frac{y-2}{2}$$

$$\Longrightarrow x - y + 3 = 0$$

put $x = t$

$y = t+3$

$z(t) = t + i(t+3) \quad -1 \le t \le 1$

Circle [parame: rep]

$x = r\cos t$

$y = r\sin t \quad 0 \le t \le 2\pi$

Ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a\cos t$

$y = b\sin t \quad 0 \le t \le 2\pi$

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$x = a\cosh t$

$y = b\sinh t \quad -\infty < t < \infty.$

3. $4x^2 + 9y^2 = 36$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

$x = 3\cos t \quad y = 2\sin t.$

$z(t) = x(t) + iy(t) = 3\cos t + i\,2\sin t$

$\cancel{0 \le t \le 2\pi}$

$z(t) = 3\cos t + i\,2\sin t \quad 0 \le t \le 2\pi$

4. $4x^2 - y^2 = 4$ through the point $(0, 2)$

$$x^2 - \frac{y^2}{4} = 1$$

$x = \cosh t \qquad y = 2\sinh t$

$$Z(t) = x(t) + i\,y(t)$$

$$= \cosh t + i\,2\sinh t, \quad -\infty < t < \infty$$

5. Upper half of $|z - 4 + i| = 4$ from the point $(5, -1)$, to $(-3, -3)$

$|z - a| = r$ centre a radius r

$a = 4 - i \qquad r = 4$

$= (4, -1).$

$$(x-4)^2 + (y+1)^2 = 4^2$$

$x - 4 = r\cos t$

$x - 4 = 4\cos t \implies x = 4 + 4\cos t$

$y + 1 = 4\sin t \implies y = -1 + 4\sin t$

$$Z(t) = (4\cos t + 4) + i\,[-1 + 4\sin t]$$

$$0 < t < \pi$$

SECOND EVALUATION METHOD

Complex line Integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt \qquad \dot{z}(t) = \frac{dz}{dt}$$

1. Evaluate $\int_C \bar{z}\, dz$: $x = 3t$ $y = t^2$
$-1 \le t \le 4$

$z(t) = x(t) + i\,y(t)$

$= 3t + it^2 \qquad -1 \le t \le 4$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, \dot{z}(t)\,dt$$

$$= \int_{-1}^{4} \overline{z(t)} \cdot \dot{z}(t)\,dt$$

$\dot{z}(t) = 3 + 2it$

$$\int_C f(z)\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt$$

$= 195 + i65.$

2. $\int_C Re(z)\,dz$ where C is a straight line

0 to $1+2i$ ie $(0,0)$ to $(1,2)$

$\frac{x}{1} = \frac{y}{2} \qquad x = t \quad y = 2t$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, \dot{z}(t)\,dt$$

$z(t) = t + i2t \qquad 0 \le t \le 1$

$\dot{z}(t) = 1 + 2i$

$$\int_C f(z)\,dz = \int_a^b f(z(t)) \cdot \dot{z}(t)\,dt \qquad Re(z) = Re(z(t))$$

$$= \int_0^1 Re(z(t)) \cdot \dot{z}(t)\,dt$$

$$= \int_0^1 t \cdot (1+2i)\,dt = \frac{1}{2} + i$$

3. Evaluate $\int_C |z|\,dz$ where:

i) C is the line segment joining $-i$ & i

ii) C is the unit circle in the left of half plane.

i) C: $-i$ to i $\quad (0,-1)$ to $(0,1)$

$$\frac{x}{0} = \frac{y+1}{2} = t \qquad -1 \leq y \leq 1$$

$$x = 0, \quad y = 2t-1 \qquad 0 \leq t \leq 1$$

$$z(t) = i(2t-1) \qquad y = -1 \Rightarrow t = 0$$

$$y = 1 \Rightarrow t = 1$$

$$\int_C f(z)\,dz = \int_0^1 |z(t)|\,\dot{z}(t)\,dt \qquad \dot{z}(t) = 2i$$

$$= \int_0^1 \sqrt{(2t-1)^2} \cdot \overset{2i}{\cancel{i(2t-1)}}\,dt$$

$$= 2i\int_0^1 (2t-1)\,dt = \underline{\underline{0}}$$

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ii) $x = \cos t \quad y = \sin t \quad z(t) = \cos t + i\sin t$

$\dot{z}(t) = -\sin t + i\cos t$

$$\int_C |z|\,dz = \int_{\pi/2}^{3\pi/2} |z(t)|\,\dot{z}(t)\,dt$$

$$= \int_{\pi/2}^{3\pi/2} \sqrt{\cos^2 t + \sin^2 t}\,(-\sin t + i\cos t)\,dt$$

$$= \left[\cos t + i\sin t\right]_{\pi/2}^{3\pi/2}$$

$$= \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) - \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \left[(0 - i) - (0 + i)\right] = -2i$$

4. Evaluate $\int_C (2+z)^2\,dz$ where C is any path connecting the points -2 & $-2+i$

$(-2, 0) \quad (-2, 1)$

$$\frac{x+2}{0} = \frac{y}{1} = t$$

$x + 2 = 0 \Rightarrow x = -2$

$y = t$

$z(t) = -2 + it$

$0 \le t \le 1$

$\dot{z}(t) = 0 + i$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt$$

$$= \int_0^1 [2+z(t)]^2 \cdot i\,dt$$

$$= i\int_0^1 [2+-2+it]^2\,dt = -i\int_0^1 t^2\,dt$$

$$= \underline{\underline{\frac{-i}{3}}}$$

5. Evaluate $\int_C Re(\bar{z}^2)\,dz$ where

i) C is the line segment joining the points $(0,0)$ & $(2,4)$

(ii) x axis from 0 to 2 & then vertically to $2+4i$

i) $(0,0)$ to $(2,4)$

$$\frac{x}{2} = \frac{y}{4} \implies x = \frac{y}{2} = t$$

$$x = t \qquad y = 2t$$

$$z(t) = t + i2t \qquad 0 \le t \le 2$$

$$\dot{z}(t) = 1 + 2i$$

$$\int_C f(z)\,dz = \int_C Re\,(z(t)^2)\cdot \dot{z}(t)\,dt$$

$$= \int_0^2 (-3t^2)(1+2i)\,dt \;=(1+2i)3\int_0^2 t^2\,dt$$

$$= -8(1+2i)$$

ii) $$\int_C \mathrm{Re}\, z^2\,dz = \int_{C_1} \mathrm{Re}\,(z^2)\,dz + \int_{C_2} \mathrm{Re}\,(z^2)\,dz.$$

C_1 $(0,0)$ to $(2,0)$

$$\frac{x}{2} = \frac{y}{0} = t \qquad x = 2t \quad y = 0$$

$$z(t) = 2t \qquad \dot{z}(t) = 2 \qquad x=0 \Rightarrow t=0$$

$$x = 2 \Rightarrow t=1$$

$$\int_{C_1} \mathrm{Re}\,(z^2)\,dz = \int_0^1 \mathrm{Re}\,(z(t)^2)\,\dot{z}(t)\,dt$$

$$= \int_0^1 4t^2 \cdot 2\,dt = \frac{8}{3}$$

C_2 $(2,0)$ to $(2,4)$

$$\frac{x-2}{0} = \frac{y}{4} = t$$

$$y=0 \Rightarrow t=0$$

$$x-2=0 \Rightarrow x=2, \qquad y=4 \Rightarrow t=1$$

$$y = 4t \qquad z(t) = 2+i4t$$

$$\dot{z}(t) = 4i$$

~~$z(t)$~~

$$\int_{C_2} Re(z^2)\,dt = \int_0^1 Re(z(t)^2)\,\dot{z}(t)\,dt$$

$$= \int_0^1 (4-16t^2)\, 4i\, dt$$

$$= 16i \int_0^1 1-4t^2\, dt$$

$$= 16i \left[t - \frac{4t^3}{3}\right]_0^1$$

$$= 16i\left[1-\frac{4}{3}\right] = \frac{-16i}{3}$$

$$\int_C Re(z^2)\,dz = \int_{C_1} + \int_{C_2} = \underline{\underline{\frac{8}{3} - \frac{16i}{3}}}.$$

6 Prove that $\displaystyle\int_C (z-z_0)^m\,dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \end{cases}$

where C is a circle with radius ρ & centre at z_0 oriented counter clockwise direction

$$|z-z_0| = \rho$$

$$z - z_0 = \rho e^{it} \implies z = z_0 + \rho e^{it}$$

$$z(t) = z_0 + \rho e^{it}$$

$$\dot{z}(t) = i\rho e^{it} \qquad 0 \le t \le 2\pi$$

$$\int_C (z-z_0)^m \, dz = \int_0^{2\pi} (z(t)-z_0)^m \, \dot{z}(t) \, dt$$

$$= \int_0^{2\pi} (z_0+\rho e^{it}-z_0)^m \, \dot{z}(t) dt$$

$$= \int_0^{2\pi} \rho^m e^{imt} \, i \rho e^{it} \, dt$$

$$= \int_0^{2\pi} i \rho^{m+1} \, e^{i(m+1)t} \, dt$$

Case-I if $m = -1$

$$\int_C (z-z_0)^m \, dz = \int_0^{2\pi} i \, dt = \underline{\underline{2\pi i}}$$

Case-II If $m \neq -1$

$$\int_C (z-z_0)^m \, dz = \int_0^{2\pi} i \rho^{m+1} \, e^{i(m+1)t} \, dt$$

$$= i\rho^{m+1} \left[e^{2\pi i(m+1)} - 1 \right]$$

$$= 0$$

$$e^{i(m+1)2\pi} = \cos(m+1)2\pi + i\sin(m+1)2\pi$$

$$= \underline{\underline{1}}$$

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Cauchy's Integral theorem

If $f(z)$ is analytic in a simply connected domain D then for every closed path in D

$$\int_C f(z)\,dz = 0$$

Eg: $\int_C e^z\,dz = 0$ for any closed path C since e^z is analytic everywhere.

1. $\int_C \frac{z}{z-2}\,dz$ $\quad C: |z| = 1$

$z - 2 = 0$ ie $z = 2$ is the singular pt [pt at which function is not analytic]

$z = 2$ is a singular point which lies outside C

$\therefore \int_C \frac{z}{z-2}\,dz = 0$ (By cauchy's Integral theorem)

2. $\int_C \frac{z^2+5}{z-3}\,dz$ $\quad C: |z-i| = 1$

$z - 3 = 0 \Rightarrow z = 3$ is the singular point which lies outside C.

$|3-i| = \sqrt{3^2+1^2}$
$= \sqrt{10} > 1$

$$\int_C \frac{z^2+5}{z-3}\,dz = 0$$

Cauchy's Integral formula

Let $f(z)$ be analytic in a simply connected domain D then for any point z_0 in D & for any closed path C in D that encloses z_0 then

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

1. $\displaystyle\int_C \frac{3z^2+7z+1}{z+1}\,dz \qquad c: |z+i| = 1$

$z+1=0 \implies z=-1$ is a sg pt

$|-1+i| = \sqrt{1^2+1^2} = \sqrt{2} > 1$

$z=-1$ is a sg pt that lies outside C

$$\int_C \frac{3z^2+7z+1}{z+1}\,dz = 0$$

2. $\displaystyle\int_C \frac{3z^2+7z+1}{z+1}\,dz \qquad c: |z+1| = 1$

$z=-1$ is a sg pt

$|-1+1| = 0 < 1$

$z=-1$ is a sg pt which lies inside the curve C.

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$z_0 = -1$

$$f(z) = 3z^2 + 7z + 1$$

$$f(z_0) = f(-1) = 3 - 7 + 1 = -3$$

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0) = \underline{-6\pi i}.$$

3. $\displaystyle\int_C \frac{1}{4z-1}\, dz \qquad C : |z| = 1$

$$= \int_C \frac{1}{4(z - \frac{1}{4})}\, dz \qquad \left|\frac{1}{4}\right| = \frac{1}{4} < 1$$

$z - \frac{1}{4} = 0 \implies z = \frac{1}{4}$ is the singular point which lies inside C

$$f(z) = 1 \implies f(z_0) = f(\tfrac{1}{4}) = 1$$

$$\int_C \frac{1}{4(z - \frac{1}{4})}\, dz = \frac{1}{4} \cdot 2\pi i\, f(z_0) = \underline{\underline{\frac{\pi i}{2}}}$$

4. $\displaystyle\int_C \frac{3z^2 + z}{z^2 - 1}\, dz \qquad C : |z - 1| = 1$

$z^2 - 1 = 0 \implies z^2 = 1$

$z = \pm 1$ are the singular points

$z = 1 \implies |1 - 1| = 0 < 1$ inside.

$z = -1 \implies |-1 - 1| = |-2| = 2 > 1$ outside.

$$\int_C \frac{3z^2 + z}{z^2 - 1}\, dz = \int_C \frac{3z^2 + z}{(z-1)(z+1)}\, dz.$$

$$= \int_C \frac{3z^2+z/z+1}{z-1}\,dz = 2\pi i\, f(z_0)$$

$$= 2\pi i \times 2$$

$$= \underline{\underline{4\pi i}}$$

$$f(z) = \frac{3z^2+z}{z+1}$$

$$f(z_0) = f(1)$$

$$= \underline{\underline{2}}$$

5. $$\int_C \frac{1}{z^2+4}\,dz \qquad C: 4x^2+(y-2)^2 = 4$$

$$z^2 = -4 \implies z = \pm\sqrt{-4}$$

$$= \pm 2i$$

$z = 2i = (0,2)$ $\quad 4\cdot 0^2 + (2-2)^2 = 0 < 4$ (inside C)

$z = -2i = (0,-2)$ $\quad 4(0) + (-2-2)^2 = 16 > 4$ (outside

$$\int_C \frac{1}{z^2+4}\,dz = \int_C \frac{1}{(z+2i)(z-2i)}\,dz$$

$$= \int_C \frac{1/z+2i}{z-2i}\,dz$$

$$f(z) = \frac{1}{z+2i}$$

$$= 2\pi i\, f(z_0)$$

$$f(z_0) = f(2i)$$

$$= 2\pi i \times \frac{1}{4i}$$

$$= \frac{1}{4i}$$

$$= \underline{\underline{\frac{\pi}{2}}}$$

DERIVATIVES OF AN ANALYTIC FUNCTION

If $f(z)$ is analytic in a domain D and has derivatives of all order which are also analytic functions in D then

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

1. Evaluate $\int_C \frac{z^2}{(z-1)^3}\,dz$ $\quad C : |z| = 2$

$z = 1$ is ~~the~~. singular point which lies inside C

$n = 3$

$$\int_C \frac{z^2}{(z-1)^3} = \frac{2\pi i}{2!} f''(z_0)$$

$z_0 = 1$

$$f(z) = z^2$$

$$f'(z) = 2z$$

$$f''(z) = 2.$$

$$f''(z_0) = f(1) = 2$$

$$\int_C \frac{z^2}{(z-1)^3}\,dz = \frac{2\pi i \times 2}{2} = \underline{\underline{2\pi i}}$$

2. $\displaystyle\int_C \frac{2z+3}{z^2-z}\,dz \qquad c: |z| = \frac{1}{2}$

$z^2 - z = 0 \implies z(z-1) = 0$

$z = 0$ & $z = 1$ are the singular points which ~~lies inside~~

$z = 0$ lies inside C

$z = 1$ lies outside C

$\displaystyle\oint_C \qquad \cancel{\int_C \frac{2z+3/z+1}{z}} \qquad \int_C \frac{2z+3}{z^2-z}\,dz = \int_C \frac{2z+3}{z(z-1)}\,dz$

$\displaystyle = \int_C \frac{2z+3/z-1}{z}\,dz = 2\pi i\, f(z_0)$

$= 2\pi i \times 3 = \underline{\underline{6\pi i}}$

$z_0 = 0 \qquad f(z) = \dfrac{2z+3}{z-1}$

$f(z_0) = f(0) = \underline{\underline{-3}}$

3. $\displaystyle\int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz \qquad c: |z| = 1$

$z = \frac{\pi}{6}$ is the singular point which lies inside C

$n=3$

$\displaystyle\int_C \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz = \frac{2\pi i}{(3-1)!} f''(z_0)$

$\displaystyle = \frac{2\pi i}{2!} f''(z_0)$

$= \pi i \, f''(z_0)$

$= \pi i \, 2 \cos \frac{\pi}{3}$

$= \pi i \cdot 2 \times \frac{1}{2}$

$= \underline{\underline{\pi i}}$

$f(z) = \sin^2 z$

$f'(z) = 2 \sin z \cos z$

$= \sin 2z$

$f''(z) = 2 \cos 2z$

$f''\left(\frac{\pi}{6}\right) = f''(z_0)$

4. $\int_C \frac{\sin z}{4z+\pi} \, dz \qquad C : |z| = 1$

$= \cancel{\frac{1}{4} \int_C \frac{\sin z}{z+\frac{\pi}{4}} \, dz.}$

$z = \frac{-\pi}{4}$ is the singular point which lies inside C

$\int_C \frac{\sin z}{4z+\pi} \, dz = \frac{1}{4} \int_C \frac{\sin z}{z+\frac{\pi}{4}} \, dz.$

$= \frac{1}{4} \cdot 2\pi i \, f(z_0) = \frac{\pi i}{2} \cdot f\left(\frac{-\pi}{4}\right)$

$= \frac{\pi i}{2} \times \frac{-1}{\sqrt{2}}$

$= \underline{\underline{\frac{-\pi i}{2\sqrt{2}}}}$

$f(z) = \sin z.$

$f\left(\frac{-\pi}{4}\right) = \sin\left(\frac{-\pi}{4}\right) = -\sin \frac{\pi}{4} = \frac{-1}{\sqrt{2}}$

5. $\int_C \frac{1}{z^2-1} \, dz \qquad C : |z| = \frac{3}{2}$

$z^2 = 1 \qquad z = \pm 1$ are the singular points which lies inside C

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$1 = A(z+1) + B(z-1)$$

$$z = 1 \Longrightarrow A = 1/2.$$

$$z = -1 \Longrightarrow B = -1/2$$

$$\frac{1}{(z-1)(z+2)} = \frac{1/2}{z-1} + \frac{-1/2}{z+1}$$

$$\int_C \frac{1}{(z-1)(z+2)}\,dz = \frac{1}{2}\int \frac{1}{z-1}\,dz - \frac{1}{2}\int_C \frac{1}{z+1}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i\, f(1) - \frac{1}{2}\cdot 2\pi i\, f(-1)$$

$$= \pi i \times 1 - \pi i \times 1 = \underline{\underline{0}}$$

6. Using Cauchy's Integral formula. Evaluate

$$\int_C \frac{z+1}{z^4+2iz^3}\,dz \quad ; \quad c : |z| = 1$$

$$z^4 + 2iz^3 = 0 \Longrightarrow z^3(z+2i) = 0$$

$\Longrightarrow z = 0,\ z = -2i$ are the singular points

$z = 0$ is the singular pt that lies inside C

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$$\int_C \frac{z+1}{z^4+2iz^3}\,dz = \int_C \frac{z+1}{z^3(z+2i)}\,dz$$

$$= \int_C \frac{z+1/z+2i}{z^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

$z_0 = 0$

$$f(z) = \frac{z+1}{z+2i}$$

$$f'(z) = \frac{2i-1}{(z+2i)^2}$$

$$\int_C \frac{z+1}{z^4+2iz^3}\,dz = \frac{\pi i\,(2i-1)}{4i}$$

$$f''(z) = \frac{2(1-2i)}{(z+2i)^3}$$

$$= \frac{\pi}{4}(2i-1)$$

$$f''(z_0) = f''(0)$$

$$= \frac{-1+2i}{4i}$$

7. $\int_C \frac{e^{-3\pi z}}{2z+i}\,dz$ C is the boundary of the square with vertices $\pm 1, \pm i$

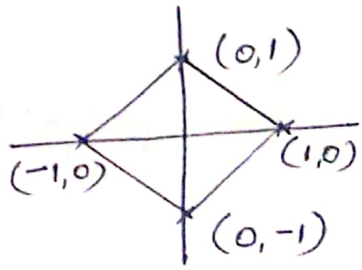
$1 \to (1,0)$

$-1 \to (-1,0)$

$i \to (0,1)$

$-i \to (0,-1)$

$$\int_C \frac{e^{-3\pi z}}{2z+i}\,dz = \frac{1}{2}\int_C \frac{e^{-3\pi z}}{z+\frac{i}{2}}\,dz$$



$z+\frac{i}{2} = 0 \implies z = -\frac{i}{2} \implies (0, \frac{-1}{2})$

is the singular point which lies inside C.

$$\int_C \frac{e^{-3\pi z}}{2z+i}\,dz = \frac{1}{2}\cdot 2\pi i\, f(z_0)$$

$$= \pi i\, e^{\frac{3\pi i}{2}} = \pi.$$

$f(z) = e^{-3\pi z}$

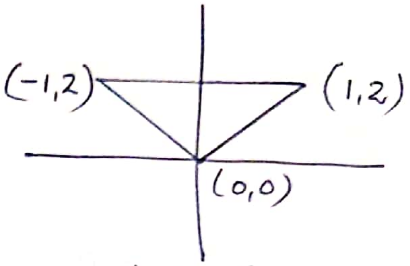
$f(z_0) = f(-\frac{i}{2})$

$= e^{\frac{3\pi i}{2}}$

$\left[e^{\frac{3\pi i}{2}} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 - i = -i\right]$

8. $\displaystyle\int_C \frac{\tan z}{z-i}\,dz$ where C is the triangle with vertices
$0,\ 1+2i,\ -1+2i$
$(0,0)\ (1,2),\ (-1,2)$

$$= \int_C \frac{\sin z}{(z-i)(\cos z)}\,dz$$



$\cos z = 0 \Rightarrow z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$ & $z = i$ are the singular points

$z = i$ lies inside C. & $\mp\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ lies outside C

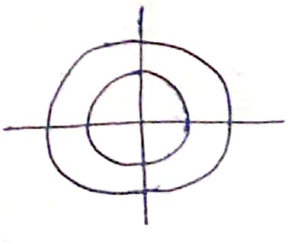
$$\int_C \frac{\tan z}{z-i}\,dz = 2\pi i\, f(z_0) = \underline{\underline{2\pi i \tan i}}$$

$f(z) = \tan z$

9. $\displaystyle\int_C \frac{4z^3-6}{z[z-1-i]^2}\,dz$ $\quad C: |z| = 3$ counter clockwise
and $|z| = 1$ clock wise

$z[z-(1+i)] = 0 \Longrightarrow z = 0$ lies outside C
$z = 1+i$ lies inside C

$$\int_C \frac{4z^3-6/z}{[z-(1+i)]^2}\,dz = \frac{2\pi i}{1!} f'(z_0)$$



$|1+i| = \sqrt{2} > 1$
$\sqrt{2} < 3$

$$f(z) = \frac{4z^3-6}{z}$$
$$= 4z^2 - \frac{6}{z}$$
$$f'(z) = 8z + \frac{6}{z^2}$$
$$f'(z_0) = f'(1+i) = 8(1+i) + \frac{6}{(1+i)^2}$$

$$= 8(1+i) + \frac{6}{2i} = 8+8i-3i = 8+5i$$

$$\int_C \frac{4z^3-6}{z(z-1-i)}\,dz = \underline{\underline{2\pi i\,(8+5i)}}$$

10. Evaluate using Cauchy's Integral formula

$\int_C \frac{z^2}{z^3-z^2-z+1}\,dz$ where C is taken counter clockwise i) $|z+1| = \frac{3}{2}$ ii) $|z-1-i| = \frac{\pi}{2}$

$z^3 - z^2 - z + 1 = 0$

$z = 1$ is a singular point

1	1	-1	-1	1
	0	1	0	-1
	1	0	-1	$\boxed{0}$

$z^2 - 1 = 0 \Rightarrow z = \pm 1$

$(z-1)(z-1)(z+1)$

$= (z-1)^2(z+1)$

i) $|z+1| = \frac{3}{2}$

$z = 1$ is a singular point which lies outside C $\left[\,|1+1| = 2 > \frac{3}{2}\right]$

$z = -1$ is the singular point which lies inside C $\left[\,|-1+1| = 0 < \frac{3}{2}\right]$

$$\int_C \frac{z^2/(z-1)^2}{z+1}\,dz = 2\pi i\; f(z_0) = 2\pi i \times \frac{1}{4}$$

$$= \underline{\underline{\frac{\pi}{2} i}}$$

ii) $z=1 \Rightarrow$ inside C

$|1-i-i| = 1 < \frac{\pi}{2}$

$f(z) = \frac{z^2}{z+1}$

$z=-1$ lies outside C

$f'(z) = \frac{2z(z+1)-z^2}{(z+1)^2}$

$|-1-1-i| = \sqrt{5} > \frac{\pi}{2}$

$f'(1) = \frac{3}{4}$

$$\int_C \frac{z^2/z+1}{(z-1)^2} dz = \frac{2\pi i}{(2-1)!} f'(z_0)$$

$$= \frac{3}{2}\pi i$$

11 $$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz \qquad c: |z| = 3$$

$z=1$ is the singular point which lies inside C & $z=2$ is the singular point which lies inside C

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

put $z=1$ $\quad 1 = -A \Rightarrow A = -1$

put $z=2$

$1 = B \Rightarrow B = 1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$= -2\pi i f(z_0) + 2\pi i f(z_0)$$

$$= -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i [\sin \pi + \cos \pi] + 2\pi i [\sin 4\pi + \cos 4\pi]$$

$$= -2\pi i (0-1) + 2\pi i [0+1] = \underline{\underline{4\pi i}}$$

12. $\int_C \frac{\tan \pi z}{(z-1)^2} dz \qquad C : |z-1| = 0.1$

$z-1 = 0 \implies z = 1, \cancel{1}$ are the singular points that lies inside C

$$\int_C \frac{\sin \pi z}{(z-1)^2 \cos \pi z} dz$$

$\cos \pi z = 0 \implies \pi z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

$\implies z = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$ these are the Sg points that lies outside C.

$z = 1$ is the singular pt that lies inside C.

$$\int_C \frac{\tan \pi z}{(z-1)^2} dz = \frac{2\pi i}{1!} f'(z_0)$$

$$= 2\pi i \times \pi$$

$$= \underline{\underline{2\pi^2 i}}$$

$f(z) = \tan \pi z$

$f'(z) = \pi \sec^2 \pi z$

$f'(1) = \pi$

13) $\displaystyle\int_C \frac{5z+7}{z^2+2z-3}\,dz \qquad C: |z-2|=2$

$z^2+2z-3=0 \implies z=1,-3$

$z=1$ is the singular pt that lies inside C

$z=-3$ " " outside C

$$\int_C \frac{5z+7/z+3}{z-1}\,dz = 2\pi i\, f(z_0) \qquad f(z)=\frac{5z+7}{z+3}$$

$$= 2\pi i\, f(1)$$

$$= 2\pi i \times 3 \qquad f(1)=\frac{12}{4}=3$$

$$= \underline{\underline{6\pi i}}$$

TAYLOR AND MACLAURIN SERIES

Taylor's theorem

Let $f(z)$ be analytic in a domain D and let $z=z_0$ be any point in D. Then there exists precisely one Taylor series

$$f(z) = f(z_0) + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^n(z_0) + R_n(z)$$

with centre z_0 that represents $f(z)$. This representation is valid in the largest open disk with centre z_0 in which $f(z)$ is analytic.

Taylor series with centre $z_0=0$ is called

Maclaurin series.

$$f(z) = f(z_0) + \frac{z}{1!} f'(z_0) + \frac{z^2}{2!} f''(z_0) + \cdots$$

The Taylor's series can also be represented as $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_n = \frac{1}{n!} f^n(z_0)$

The circle $|z - z_0| = R$ is called the circle of convergence. and its radius R is the radius of convergence.

I. $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$ valid in $|z| < 1$

$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$ "

$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$ "

$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$ "

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ $|z| < \infty$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ "

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$ "

$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ "

$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ "

$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$ "

Note: The radius of convergence is the shortest distance from z_0 to the nearest singular point of $f(z)$

1. Find the Taylor series expansion of $f(z) = \frac{1}{z}$ at $z = i$

$z_0 = i$

$f(z) = \frac{1}{z}$ $\qquad$ $f(z_0) = \frac{1}{i}$

$f'(z) = \frac{-1}{z^2}$ $\qquad$ $f'(z_0) = \frac{-1}{-1} = 1$

$f''(z) = \frac{2}{z^3}$ $\qquad$ $f''(z_0) = \frac{2}{-i} = \frac{-2}{i}$

$$f(z) = f(z_0) + \frac{f'(z_0)(z-z_0)}{1!} + \frac{(z-z_0)^2 f''(z_0)}{2!} + \cdots$$

$$= \frac{1}{i} + \frac{1(z-i)}{1} + \frac{-2}{i}\frac{(z-i)^2}{2!} + \cdots$$

2. Find the Taylor's series expansion of $f(z) = \frac{\sin z}{z-\pi}$ about $z = \pi$

$$f(z) = \frac{\sin z}{z-\pi} = \frac{\sin(z-\pi+\pi)}{z-\pi} = \frac{-\sin(z-\pi)}{z-\pi}$$

$$= \frac{-1}{z-\pi}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} - \cdots\right]$$

$$= -\left[1 - \frac{(z-\pi)^2}{3!} + \frac{(z-\pi)^4}{5!} - \cdots\right]$$

3. Find the Taylor series of $f(z) = \frac{1}{1+z}$ with center $z_0 = -i$

$$\frac{1}{1+z} = \frac{1}{1+z+i-i} = \frac{1}{1-i+z+i}$$

$$= \frac{1}{(1-i)\left[1+\frac{z+i}{1-i}\right]} = \frac{1}{1-i}\left[1+\frac{z+i}{1-i}\right]^{-1}$$

$$= \frac{1}{1-i}\left[1-\left(\frac{z+i}{1-i}\right)+\left(\frac{z+i}{1-i}\right)^2 - \cdots\cdots\right]$$

$$= \left[\frac{1}{1-i} - \frac{z+i}{(1-i)^2} + \frac{(z+i)^2}{(1-i)^3} - \cdots\cdots\right]$$

4. Expand the function $f(z) = \frac{1}{z}$ in Taylor's series with center $z_0 = i$. Find the radius of convergence.

$$\frac{1}{z} = \frac{1}{z-i+i} = \frac{1}{i}\left(1+\frac{z-i}{i}\right) = \frac{1}{i}\left[1+\frac{z-i}{i}\right]^{-1}$$

$$= \frac{1}{i}\left[1 - \frac{z-i}{i} + \left(\frac{z-i}{i}\right)^2 - \left(\frac{z-i}{i}\right)^3 + \cdots\cdots\right]$$

$$= \frac{1}{i} + z-i - \frac{(z-i)}{i} - (z-i)^3 + \cdots$$

Radius of convergence = shortest distance from z_0 to the nearest singular point of $f(z)$

$$R = \sqrt{(i-0)^2} = \underline{\underline{i}}$$